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FURTHER MODELING OF TURBULENT WALL PRESSURE ON A CYLINDER FOR A--ETC(U)

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FURTHER MODELING OF TURBULENT WALL PRESSURE ON A CYLINDER FOR ARBITRARY BOUNDARY-LAYER THICKNESS

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ABSTRACT

The relation of the harmonic spectral densities $P_m(k,\omega)$ of wall pressure on a cylinder in incompressible parallel flow to pure turbulent-stress sources at highly nonconvective (low) wavenumbers ($|k|U_\infty/\omega \ll 1$) is formulated for arbitrarily large values of $|k|a$ and $|k|\delta$, as needed in application to towed-array noise. For the axisymmetric component of actual concern, the result to zero order in kU_∞/ω is used in conjunction with model properties of the source spectra assumed in earlier work to characterize the dependence of the wall-pressure spectrum $p_0(k,\omega)$ where $|k|\delta \gtrsim 1$. Where $|k|a \lesssim 1$, the zero-wavenumber result previously found for $|k|\delta \lesssim 1$ still applies, and where $|k|a \gtrsim 1$, an additional factor $\sim(ka)^2$ enters. The results tend to confirm, within the uncertainty of the source modeling, the kind of multiplicative dependence of $p_0(k,\omega)$ on ka embodied in the current semiempirical model.

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1. INTRODUCTION

Theoretical modeling of the spectral density $P_0(k,\omega)$ of the axisymmetric component of wall pressure on a cylinder in parallel flow was pursued in Ref. 1. This was done in a manner that, though still essentially dependent on *ad hoc* modeling of certain spectra of the turbulence field, provides a more uniform framework and guide than that used, for example, in Ref. 2. Specifically, the subject spectrum $P_0(k,\omega)$ was related to spectra of turbulent velocity products (or Reynolds stresses), rather than partly to these and partly to a spectrum of the wall normal component of velocity (first power, not a velocity product). This new relation, in general, exhibits rather than obscures the wave structure inherent in the pertinent dynamic equations.*

The explicit relation of wall pressure to turbulent stress spectra applied in Ref. 1 was derived in Ref. 3 for a certain domain of low wavenumbers, defined roughly by the conditions

$$|k|U_\infty/\omega \ll 1, \qquad |k|\delta \ll 1 , \tag{1-1}$$

where δ is a boundary-layer thickness outside which fluctuations are considered negligible. The main purpose of the present memo

*At the same time, it is recognized, the wave structure has no *direct* relevance at subconvective wavenumbers, and the convective domain, on the other hand, cannot be dealt with explicitly without invocation of a closure assumption in some form.

is merely to extend the relation to a domain omitting the second of conditions (1) and to obtain generalized results for this extended domain by use of the same type of source model principally entertained in Ref. 1. In much of the parameter domain of interest with regard to towed-array noise, one has $|k|U_\infty/\omega \ll 1$ (i.e. subconvective wavenumbers) but $|k|\delta \gg 1$, so that the present extension is needed.

In Ref. 1, notwithstanding the first of restrictions (1), the relation for wall pressure was applied with source spectra modeled as though results should be applicable even in the convective domain. This was done to furnish possibly indicative comparison with extant semiempirical models of the wall pressure that encompass the convective domain.* Source spectra are likewise modeled here in this same generalized form.

In Sec. 2 the perturbation expansion of wall pressure based on only the first of conditions (1) is formulated, and the zero-order result given. In Sec. 3 results for the wall-pressure spectrum $P_0(k,\omega)$ are obtained in this zero-order approximation from the intended type of source model, and the implication drawn for current modeling of towed-array wall pressure. In an appendix the next-order contribution to wall pressure is derived.

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*In the "mixed-source" formulation of Ref. 3 (based on the Poisson equation for pressure in the Lighthill formulation for incompressive flow), the formal relation of pressure to sources is readily available without even the first of restrictions (1), and the problem of treating the critical layer via closure assumptions that enters the "pure-source" formulation when applied to the convective domain is absent. Presumably, however, this absence is no virtue but implies sweeping the dynamic structure totally under the rug (cf. preceding footnote).

2. THE RELATION OF WALL PRESSURE TO FLUCTUATING STRESSES AT WAVENUMBERS WELL BELOW CONVECTIVE BUT POSSIBLY ABOVE THE RECIPROCAL DIAMETER OR LAYER THICKNESS

In Ref. 3 a formal solution was obtained for the wall pressure in inviscid incompressible flow along a cylinder.* The subsequent expansion procedure was valid provided that conditions (1-1) are satisfied. Beginning with the same formal solution for the wall pressure (Ref. 3, Sec. 5), a modification of the scaling used in that development can be made which removes the second restriction on k. The resulting expansion in powers of $\varepsilon \equiv kU_\infty/\omega$ of the transformed pressure amplitude $\hat{p}_m(k,\omega)$, though somewhat more cumbersome, is valid for arbitrary values of $k\delta$. Since the method to be presented closely parallels the procedure of Ref. 3, only a brief outline of the results will be recorded here. Furthermore, explicit solutions will be given only for the axisymmetric component (m=0) of the wall pressure, though the method is applicable for any circumferential mode.

The transformed pressure amplitudes $\hat{p}_m(r,k,\omega)$ of the circumferential modes $e^{im\theta}$ are related to source function $\hat{s}_m$, which contains components of the fluctuating stress tensor, via the ordinary differential equation

*This was done also for slightly compressible flow, but that aspect is not of concern here.

$$(\omega - kU)\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\hat{p}_m}{dr}\right) - \left(\frac{m^2}{r^2} + k^2\right)\hat{p}_m\right] + 2kU'\frac{d\hat{p}_m}{dr} = \rho\hat{s}_m. \qquad (2\text{-}1)$$

The explicit form of $\hat{s}_m$ is given in Ref. 3, Eqs. 37 and 38. The formal solution of (2-1), subject to the boundary conditions

$$|\hat{p}_m(r)| < \infty \text{ at } r = \infty$$

and

$$|\hat{p}_m'(r)| = 0 \text{ at } r = a ,$$

the cylinder radius, may be obtained in a straightforward manner. When evaluated at r=a, the axisymmetric component becomes

$$\hat{p}_0(a,k,\omega)/\rho = \Delta^{-1}\Gamma_c(a)\int_a^\infty dr\left[(\hat{T}_{rr}-\hat{T}_{xx})k^2 r\phi E_c + (\hat{T}_{rr}+\hat{T}_{\theta\theta})\phi' E_c - (T_{rr}-T_{\theta\theta}+i2krT_{rx})\phi E_c'(r)\right] \qquad (2\text{-}2)$$

where

$$\Gamma_c(s) = \phi\overline{\Gamma}(r)/\omega$$

$$E_c(r) = \phi\overline{E}(r)/\omega$$

$$\Delta = r(E_c\Gamma_c' - \Gamma_c E_c')$$

$$\phi = (1-kU/\omega)^{-1} ,$$

and $\overline{E}$ and $\overline{\Gamma}$ are the solutions of the homogeneous form of (2-1), satisfying the boundary conditions

$$|\overline{E}(r)| < \infty \quad \text{at } r = \infty$$

$$|\overline{\Gamma}'(r)| = 0 \text{ at } r = a . \qquad (2\text{-}3)$$

A low-wavenumber expansion for $\hat{p}_0$ requires corresponding expansions for the homogeneous solutions $E_c(r)$ and $\Gamma_c(r)$. These may be obtained more conveniently by considering a self-adjoint form of (2-1),

$$\frac{d^2 g_m}{dr^2} - \left[k^2 + \frac{m^2}{r^2} + \frac{(r^{\frac{1}{2}}\phi)''}{r^{\frac{1}{2}}\phi}\right] g_m = 0 \tag{2-4}$$

where

$$g_m(r) = \hat{p}_m(r) r^{\frac{1}{2}}/(\omega - kU) \ .$$

Adopting the scales $\xi = r/\delta$, $U = U_\infty V(\xi)$, $\varepsilon = kU_\infty/\omega$ and $s = k\delta$, (2-4) becomes

$$\frac{d^2 g_m}{d\xi^2} - \left[s^2 + \frac{m^2 - 1/4}{\xi^2} + \frac{\varepsilon V'}{\xi(1-\varepsilon V)} + \frac{\varepsilon V''}{1-\varepsilon V} + \frac{2\varepsilon^2 V'^2}{(1-\varepsilon V)^2}\right] g_m = 0 \ , \tag{2-5}$$

which, when expanded to second order in ε, yields

$$\frac{d^2 g_m}{d\xi^2} - \left\{s^2 + \frac{m^2 - 1/4}{\xi^2} + \varepsilon\left(V'' + \frac{V'}{\xi}\right) + \varepsilon^2\left[2V'^2 + V\left(V'' + \frac{V'}{\xi}\right)\right]\right\} g_m = 0. \tag{2-6}$$

Inserting a power series representation for g_m,

$$g_m = g_0 + g_1 + \varepsilon^2 g_2 + \ldots$$

results in the following hierachy of equations

$$\varepsilon^0: \quad g_0'' - [s^2 + (m^2 - 1/4)/\xi^2] g_0 = 0$$

$$\varepsilon^1: \quad g_1'' - [s^2 + (m^2 - 1/4)/\xi^2] g_1 = g_0 (V'' + V'/\xi)$$

$$\varepsilon^{2}" \quad g_2'' - [s^2 + (m^2 - 1/4)/\xi^2] g_2 = g_1 (V'' + V'/\xi) + g_0 [2V'^2 + V(V'' + V'/\xi)]. \tag{2-7}$$

The boundary conditions identifying the two solutions $\bar{E}$ and $\bar{\Gamma}$ must also be expanded in powers of ε. The wall condition $p'_m(a) = 0$, corresponding to $\bar{\Gamma}$, can be written

$$\frac{d}{d\xi}\left[(1-\varepsilon V(\xi))\xi^{-\frac{1}{2}}g_m(\xi)\right] = 0 \quad \text{at} \quad \xi = a/\delta \equiv \xi_0 ,$$

which, when expanded, supplies a boundary condition for each of the equations in (2-7)

$$\begin{aligned}
\varepsilon^0:&\quad g'_0(\xi_0) = (1/2)\xi_0^{-1}g_0 \\
\varepsilon^1:&\quad g'_1(\xi_0) = g_0 V'(\xi_0) + \tfrac{1}{2}\xi_0^{-1}g_1 \\
\varepsilon^2:&\quad g'_2(\xi_0) = g_1 V'(\xi_0) + \tfrac{1}{2}\xi_0^{-1}g_2 \qquad \text{etc.}
\end{aligned} \tag{2-8}$$

The boundedness condition on $p_m(r)$ at large distances from the cylinder translates immediately into the condition $\xi^{-\frac{1}{2}}\, g_m(\xi)<\infty$ as $\xi\to\infty$. It is more convenient, however, to form an equivalent condition which is applied at $\xi = 1$. Outside the boundary layer, where $\xi\geq 1$, (2-5) simplifies to

$$\frac{d^2 g_m}{d\xi^2} - \left[s^2+(m^2-1/4)/\xi^2\right]g_m = 0 ,$$

which has as its finite solution

$$g_m(\xi) = A\xi^{\frac{1}{2}}K_m(|s|\xi) . \tag{2-9}$$

The solution for $\xi<1$ must merge with (2-9) in such a way that $g'_m(\xi)/g_m(\xi)$ is continuous at $\xi=1$, or, equivalently,

$$\frac{dg_m(\xi)}{d\xi}K_m(|s|\xi) = g_m(\xi)[\tfrac{1}{2}K_m(|s|\xi)+\frac{d}{d\xi}K_m(|s|\xi)] \text{ at } \xi=1.$$

Inserting the expansion for g_m results in a sequence of boundary conditions to be applied at $\xi=1$,

$$\frac{dg_i(\xi)}{d\xi}K_m(|s|\xi) = g_i(\xi)[\tfrac{1}{2}K_m(|s|\xi)+\frac{d}{d\xi}K_m(|s|\xi)] \quad , \tag{2-10}$$

on the ε^i level of the hierarchy given in (2-7). This method of selecting the bounded solution corresponding to $\bar{E}$ proves to be more tractable when the higher-order (inhomogeneous) equations of (2-7) are considered.

The zero-order (ε^0) solutions for the axisymmetric mode (m=0) may be readily obtained by solving the first equation of (2-7) subject to (2-8) and (2-10). The solution bounded at ∞ is

$$g_0(\xi) = B_0\xi^{\frac{1}{2}}K_0(|s|\xi) \quad , \tag{2-11}$$

and

$$g_0(\xi) = A_0\xi^{\frac{1}{2}}\left[I_0(|s|\xi)-\frac{I_0'(|s|\xi_0)}{K_0'(|s|\xi_0)}K_0(|s|\xi)\right] \quad [(\ )' \equiv d/d\xi] \tag{2-12}$$

satisfies the proper wall condition at $\xi=\xi_0$. To this order, and with the normalization $A_0=B_0=1$, the functions E_c and Γ_c in (2-2) become

$$E_c(r) = K_0(|k|r)$$
$$\Gamma_c(r) = I_0(|k|r)-\frac{I_0'(|k|a)}{K_0'(|k|a)}K_0(|k|r). \quad [(\ )' \equiv d/dr]$$

This implies the following zero-order (in ε) approximation for the axisymmetric component of the wall pressure

$$\hat{p}_0(a,k,\omega)/\rho = \frac{1}{|ka|K_1(|k|a)}\int_a^\infty dr[(\hat{T}_{rr}-\hat{T}_{xx})k^2 rK_0(|k|r) +$$

$$(\hat{T}_{rr}+\hat{T}_{\theta\theta})(k/\omega)U'(r)K_0(|k|r) +$$

$$(\hat{T}_{rr}-\hat{T}_{\theta\theta}+2ikr\hat{T}_{rx})|k|K_1(|k|r)]. \qquad (2\text{-}13)$$

In the limit $k\delta$ (and ka)$\to 0$, Eq. 2-13 attains the proper form (Eq. 58, Ref. 3). It provides an expression for the wall pressure (to zeroth order in ε) having an extended domain of validity.

In the appendix it is indicated how an additional order of accuracy (in ε) may be achieved. The following modeling, however, will employ only the zero-order result (2-13).

3. THE WALL-PRESSURE SPECTRUM BASED ON ASSUMED PROPERTIES OF SOURCE SPECTRA

We wish to use the relation (2-13) of wall-pressure amplitude in conjunction with specific assumed properties of the cross-spectra $E_{ijk\ell}(r,r',k,\omega)$ of fluctuating stress components, corresponding to the product $\hat{T}^*_{ij}(r,k,\omega)\hat{T}_{k\ell}(r',k',\omega)$, to infer a resulting expression for the wall-pressure spectrum $P_0(k,\omega)$ and examine its limiting dependences. Specifically, we assume a model of the type discussed in Ref. 1 in connection with Eqs. 16 to 19, and 25. In this context, the model is exemplified by the form

$$E_{ijk\ell}(r,r',k,\omega) = c_{ij\ell m} v_*^3 (a^2/rr') L(r) L(r') (1+\alpha k_+ r)^{-\frac{1}{2}} (1+\alpha k_+ r')^{-\frac{1}{2}} \times S^*(k_+ y) S(k_+ y') \ . \tag{3-1}$$

$(r=a+y, r'=a+y')$, where the scale length L is taken to be given by the mixing-length model form (Eq. 25)

$$L(r) \sim (ar)^{\frac{1}{2}} \ell n(r/a) \ ; \tag{3-2}$$

$$k_+ \equiv [(\omega - u_c k)^2/(hv_*)^2 + k^2]^{\frac{1}{2}} \ , \tag{3-3}$$

$\alpha \sim 1$, $u_c \sim U_\infty$, the $c_{ij\ell m}$ are dimensionless constants (~ 1), and $S(k_+ y)$ is a well-behaved function that is ~ 1 for $k_+ y \lesssim 1$ and decreases rapidly for $k_+ y \gtrsim 1$. Though not expressed in the notation, $S(k_+ y)$ is assumed negligible for $y \gtrsim \delta$ independently of k_+ and likewise to vanish appropriately within the viscous sublayer as $yv_*/\nu \to 0$.

$P_0(k,\omega)$ inferred by reference to (2-13), as usual (see Ref. 1), is expressible as a sum of a number of double integrals over r and r'. Under assumption (3-1), each of these is simply a square of a single integral. To simplify expressions, temporarily imagine, for example, that $c_{ij\ell m}$ in (3-1) also had the factorable form $c_{ij}^* c_{k\ell}$. In (2-13) the mean velocity gradient outside the sublayer may be approximated as in Ref. 1, Eq. 24, by

$$U'(r) = Av_*/r\ell nr \quad (A \simeq 2.5), \qquad r-a \gtrsim 6\nu/v_* . \tag{3-4}$$

The spectral density obtained from (2-13), with tentative neglect of source strength within the sublayer, is then given by

$$P_0(k,\omega) \simeq c^2 v_*^3 a^2 I^2, \tag{3-5}$$

$$I = \frac{1}{|k|aK_1(|k|a)} \int_{6\nu/v_*}^{\infty} dyS(k_+y)[1+\alpha k_+a(1+y/a)]^{-\frac{1}{2}}$$

$$\times(1+y/a)^{-\frac{1}{2}}\ell n(1+y/a)\{(c_{rr}-c_{xx})k^2a(1+y/a)K_0{}'(|k|a(1+y/a))$$

$$+[c_{rr}-c_{\theta\theta}+i2ka(1+y/a)c_{rx}]|k|K_1(|k|a(1+y/a))$$

$$+(c_{rr}+c_{\theta\theta})k(Av_*/\omega a)(1+y/a)^{-1}[\ell n(1+y/a)]^{-1}K_0(|k|a(1+y/a))\} \tag{3-5a}$$

The essential aspects of the dependence of (3-5a) and hence $P_0(k,\omega)$ under present assumptions are demonstrated by considering the opposite limits $ka \lesssim 1$ and $ka \gtrsim 1$ where in both instances $k\delta \gtrsim 1$. [The case where $ka \lesssim 1$ and $k\delta \lesssim 1$ was already dealt with in Ref. 1 and corresponded to approximating { } in (3-5a) by the dominant limiting terms $(c_{rr}-c_{\theta\theta})|k|K_1(|k|r) \to (c_{rr}-c_{\theta\theta})r^{-1}$.]

First, when $|k|a \ll 1$ and $|k|\delta \gg 1$, since $k_+ > |k|$ and the factor $S(k_+y)$ cuts off contributions to the integral at $y \sim k_+^{-1}$, one has for values of y that may contribute appreciably: $|k|(a+y) \lesssim |k|(a+k_+^{-1}) \lesssim k_+(a+k_+^{-1}) \sim 1$. It is thus not a bad approximation to regard $|k|(a+y)$ in $K_j(|k|(a+y))$ as small. Hence, roughly the same result as the earlier one for $|k|\delta \ll 1$ is again obtained.*

---

*That result corresponds to replacing the lower limit $6\nu/v_*$ in (3-5a) by zero and hence loses validity where $6k_+\nu/v_* \gtrsim 1$. The same is true of the result given for $|k|a \gg 1$.

Next, when $|k|a>>1$, the $K_j(|k|(a+y))$ may be approximated for the entire range of integration by their limiting forms for large real argument. Eq. 3-5a then becomes

$$I \to a^{-\frac{1}{2}}\int_{6\nu/v_*}^{\delta} dyS(k_+y)\exp(-|k|y)[1+\alpha k_+a(1+y/a)]^{-\frac{1}{2}}$$

$$\times\{\ln(1+y/a)[|k|a(c_{rr}\pm i2c_{rx}-c_{xx})-(1+y/a)^{-1}(c_{\theta\theta}-c_{rr})] \tag{3-6}$$

$$\pm A(v_*/\omega a)(1+y/a)^{-2}(c_{rr}+c_{\theta\theta})\}$$

for $k\gtrless 0$, respectively. Here $Av_*/\omega a<<1$ (even if $k\sim\omega/U_\infty$, since $|k|a>>1$).* Barring unexpected cancellation among c_{ij}'s, the term $\propto c_{rr}+c_{\theta\theta}$ as well as that $\propto c_{\theta\theta}-c_{rr}$ may be neglected. Further, $S(k_+y)$ cuts off contributions (since $k_+>>\delta^{-1}$) above $y\sim k_+^{-1}\lesssim|k|^{-1}$; for contributing y, then, $y/a\lesssim 1/|k|a<<1$. Expression (3-6) then reduces, apart from a factor of the order of unity, to

$$I \sim (c_{rr}\pm i2c_{rx}-c_{xx})|k|k_+^{-5/2} \sim |k|k_+^{-5/2}$$

and (3-5), in turn, to

$$P_0(k,\omega) \sim \rho^2 v_*^3(ka)^2 k_+^{-5}. \tag{3-7}$$

Relaxing the assumption that the $c_{ijk\ell}$ in (3-1) have a product form and, in fact, generalizing $S^*(k_+y)S(k_+y')$ to become $S_{ij}^*(k_+y)S_{k\ell}(k_+y')$ (with similar properties) will evidently have no

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*More generally, for $|k|a>>1$, Eq. 2-13 becomes

$$\hat{p}_c(a,k,\omega)/\rho \to a^{-1}\int_0^\infty dy(1+y/a)^{-\frac{1}{2}}\exp(-|k|y)[(c_{rr}\pm i2c_{rx}-c_{xx})|k|(a+y)$$

$$+c_{rr}-c_{\theta\theta}\pm(c_{rr}+c_{\theta\theta})U'(a+y)/\omega]$$

effect on the functional dependences. Likewise, these dependences would prevail for a class of spectral functions somewhat more general than the product form of Eq. 3-1.

In the limit of small $|k|(a+\delta)$, $P_0(k,\omega)$, it is noted, is dominated by the contribution from the source spectral density related to $\hat{T}_{rr}-\hat{T}_{\theta\theta}$, and in the large-$|k|a$ limit to that related rather to $\hat{T}_{rr} \pm i2\hat{T}_{rx} - \hat{T}_{xx}$.

The form (3-7) agrees with the form assumed by the current semiempirical model for $P_0(k,\omega)$ (e.g., see Ref. 1, Eq. 9a) in the pertinent limit. The present results for functional dependence on ka result from the form of the homogeneous operator on pressure in the underlying equation and from the cylindrical geometry. As it turns out (and as one might hope), these results based on an underlying equation consistently involving velocity-product sources yield no departure from those obtained in Ref. 9 based on a Lighthill formulation involving mixed sources.* Hence, theoretical modeling still does not suggest that a trial form different from our current wall-pressure model given in Ref. 1, Eq. 9a (with term linear in ka included) is likely preferable.

The question of possible entrance of another length scale, as discussed in Ref. 1, and of the numerical values of parameters in the model have not been further addressed here.

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*Since Ref. 2 was written, the assumed form of the detailed dependence on k_+ has evolved to a somewhat different form, as discussed in Ref. 1, Sec. 3.

APPENDIX. EXPANSION OF WALL PRESSURE TO NEXT ORDER IN kU_∞/ω WITH $k\delta$ AND ka ARBITRARY

The components necessary for an $O(\xi)$ correction to $\hat{p}_0$, given by (2-2), are calculated in this appendix. This requires the solution of

$$g_1''(\xi)-(s^2-1/4\xi^2)g_1 = g_0[V''(\xi)+V'/\xi] \qquad \text{(A-1)}$$

subject to the boundary conditions

$$g_1'(\xi_0)-\tfrac{1}{2}\xi_0^{-1}g_1(\xi_0) = g_0(\xi_0)V'(\xi_0) \qquad \text{(A-2)}$$

and

$$g_1'(\xi)K_0(|s|\xi) = g_1(\xi)[\tfrac{1}{2}K_0(|s|\xi)+\tfrac{d}{d\xi}K_0(|s|\xi)] \quad \text{at } \xi = 1 , \qquad \text{(A-3)}$$

which yield Γ_C and E_C, respectively.

The general solution to (A-1) may be written

$$g_1(\xi) = A_1\psi_1+B_1\psi_2-\int_{\xi_0}^{\xi} [\psi_2(\xi)\psi_1(t)-\psi_1(\xi)\psi_2(t)]g_0(t)[V''(t)+V'/t]dt ,$$

where $\psi_1 = \xi^{\frac{1}{2}}I_0(|s|\xi)$, $\psi_2 = \xi^{\frac{1}{2}}K_0(|s|\xi)$ are the independent solutions of the homogeneous form of (A-1). The ratio of the constants A_1, B_1 is determined by a normalization condition, and that ratio is then fixed by the application of either (A-2) or (A-3).

Consider first the boundary condition (A-2), corresponding to the Γ solution. With $g_0(t)$ given by (2-12), (A-4) may be written

$$\xi^{-\frac{1}{2}} g_1(\xi) = I_0(|s|\xi)\left[A_1 + \int_{\xi_0}^{\xi} K_0(|s|t)\left(I_0(|s|t) - \frac{I_0'(|s|\xi_0)}{K_0'(|s|\xi_0)} K_0(|s|t)\right)(tV'(t))'dt\right.$$

$$\left. + K_0(|s|\xi)\left[B_1 - \int_{\xi_0}^{\xi} I_0(|s|t)\left(I_0(|s|t) - \frac{I_0'(|s|\xi_0)}{K_0'(|s|\xi_0)} K_0(|s|t)\right)(tV'(t))'dt\right.\right.$$

Applying the boundary condition (A-2) yields the following relation between A_1 and B_1,

$$A_1 I_0'(|s|\xi_0) + B_1 K_0'(|s|\xi_0) = -V'(\xi_0)/\xi_0 K_0'(|s|\xi_0) \ .$$

The normalization condition $g_1(\xi_0) = 0$, a choice for convenience only, provides the additional condition

$$0 = A_1 I_0(|s|\xi_0) + B_1 K_0(|s|\xi_0) \ .$$

With A_1, B_1 so determined, the first-order correction $g_1(\xi)$ becomes

$$\xi^{-\frac{1}{2}} g_1(\xi) = \frac{-V'(\xi_0)}{K_0'(|s|\xi_0)} \left[K_0(|s|\xi_0) I_0(|s|\xi) - I_0(|s|\xi_0) K_0(|s|\xi)\right]$$

$$-\int_{\xi_0}^{\xi} (tV'(t))' \left[K_0(|s|\xi) I_0(|s|t) - K_0(|s|t) I_0(|s|\xi)\right]\left[I_0(|s|t) - \frac{I_0'(|s|\xi_0)}{K_0'(|s|\xi_0)} K_0(|s|t)\right] dt$$

or, equivalently, the $O(\xi)$ correction to $\Gamma_c(r)$ in (2-2), denoted by $\Gamma_1(r)$, is

$$\Gamma_1(r) = \frac{-V'(a)}{K_0(|k|a)} \left[K_0(|k|a) I_0(|k|r) - K_0(|k|r) I_0(|k|a)\right]$$

$$-\int_a^r \frac{d}{dt}(tV'(t)) \left[K_0(|k|r) I_0(|k|t) - K_0(|k|t) I_0(|k|r)\right]\left[I_0(|k|t) - \frac{I_0'(|k|a)}{K_0'(|k|a)} K_0(|k|t)\right] dt$$

(A-5)

where $V(r) = U(r)/U_\infty$. It is readily established that in the limit $k\delta$ (and $ka)\to 0$, $\Gamma_1(r)$ given by (A-5) tends to $V(r)$ so that to $O(\varepsilon)$

$$\Gamma_c(r) \to 1+\varepsilon V(r),$$

in agreement with the results of Ref. 3.

In a similar manner, the first-order correction to $E_c(r)$ is calculated by subjecting the general form (A-4) to the appropriate boundary condition (A-3) and using $g_0(\xi)$ as given by (2-11). The normalization condition $g_1(\xi_0) = 0$ again provides the relation

$$0 = A_1 I_0(|s|\xi_0) + B_1 K_0(|s|\xi_0) \ ,$$

and (A-3) in turn yields

$$A_1 = -\int_{\xi_0}^{1} K_0^2(|s|t)(tV'(t))'dt.$$

The appropriate solution is

$$\xi^{-\frac{1}{2}}g_1(\xi) = -I_0(|s|\xi)\left[\int_{\xi}^{1} K_0^2(|s|t)(tV'(t))'dt\right]$$
$$-K_0(|s|\xi_0)\left[\int_{\xi_0}^{\xi} I_0(|s|t)K_0(|s|t)(tV')'dt-\frac{I_0(|s|\xi_0)}{K_0(|s|\xi_0)}\int_{\xi_0}^{1} K_0^2(|s|t)(tV')'dt\right]$$

which, when written as a function of r, denoted by $E_1(r)$, provides the first-order correction to $E_c(r)$,

$$E_c(r) = K_0(kr) + \varepsilon E_1(r) \ ,$$

with

$$E_1(r) = -I_0(|k|r)\int_r^\delta K_0^2(|k|t)(tV')'dt$$

$$-K_0(|k|r)\left[\int_a^r I_0(|k|t)K_0(|k|t)(tV')'dt - \frac{I_0(|k|a)}{K_0(|k|a)}\int_a^\delta K_0^2(|k|t)(tV')'dt\right]. \tag{A-6}$$

The limiting form of (A-6) when $k\delta \to 0$ properly reduces to the formula given in Reference 3, where $E_c(r)$ was calculated with $k\delta$ assumed small at the outset.

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3. D. M. Chase and C. F. Noiseux, "Turbulent wall pressure at low wavenumbers: relation to nonlinear sources in planar and cylindrical flow," *J. Acoust. Soc. Am.*, in publication, 1982.